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Completeness of P-Summable as Hilbert Space with Its Dual Space

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Abstract

Hilbert space is a complete inner product space, meaning that each Cauchy sequence converges to a point in that space. One of the vector spaces that will be examined as the inner product space is p-summable space. The inner product space is a subset of vector spaces that have special properties that must be fulfilled. One way to prove vector space is the inner product space is to use parallelogram equality theorems. After it is known that the vector space is the inner product space, the completeness of the space will be proven using the dual space. The space used is the p-summable space, data that can be changed in a sequence form will be usable in this study. The results of this study will be useful as another application in determining a Hilbert space by using a method that is different from the definition. The analysis used will show comparison of the speed of completion accuracy will be a benchmark in this study, so that will be a new reference in determining a space is Hilbert space.

***Keywords*:** Dual, Hilbert space, Isomorphism

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Introduction

The study of the proof of the Banach space is the Hilbert space using the Parallelogram Theorem has made many Banach spaces converge into Hilbert space. In addition to using the Parallelogram Theorem, the Banach space approach into a Hilbert space is to use a linear operator. The linear operator used is wise isomorphism.

The use of these operators requires a dual from Banach space, so science is needed in determining the dual. The space to be used in this research is space ℓ^p which is a p-summable sequences of real numbers. In statistics, data can be written in the form of a row consisting of dependent and independent variables, so this research can be used on the type of data that can be written in sequence. Data that can be used as an example is a regression dataset.

In (Klipfel, 2009) stated that the use of Hilbert space was on its application to quantum mechanics. Quantum mechanics is important to understand the behavior of systems on a smaller scale, meaning that behaviors in the data set can be known if the data includes the Hilbert space. The behavior of the data can indicate relationships between data.

Materials

Based on book (Kreyszig, 1978), let X is a space norm, then the set of all bounded linear functionals on X forming a norm space with norm defined by

$$\|f\| = \sup_{x\in X, \|x\|=1} |f(x)|$$

is said to be a dual space and it is denoted by X '.

The norm space to be used in this research is ℓ^p space with $p \geq 2$ which is the space of all the p-summable sequences of real numbers (Konca, Idris, & Gunawan, 2015). The use of space ℓ^p is intended to have distinction on $p = 2$ and $p > 2$.

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At (Drivaliaris & Yannakakis, 2005), it is said that let X is a normed space with norm $\|\cdot\|$ and X' is the dual space of the X and $\langle\cdot,\cdot\rangle$ is their duality product. The following definitions are explanations of the duality product, as follows:

Definition 1. T is said to be positive if

$$\langle Tx, x\rangle \geq 0, \text{ for all} x, y \in X.$$

Definition 2. T is said to be symmetric if

$$\langle Tx, y\rangle = \langle Ty, x\rangle, \text{ for all } x, y \in X.$$

Theorem 3. A Banach space is isomorphic into a Hilbert space if and only if there is an isomorphism T of X onto X' in such a way that

$$m\|x\|^2 \leq \langle Tx, x\rangle \leq M\|x\|^2, \text{ for all } x \in X,$$

where m and M are positive constants.

Isomorphism is a normed space X onto a normed space X' is a bijective linear operator, $T: X \rightarrow X'$, which is a norm, so for every $x \in X$ applies(Šemrl, 2008)

$$\|Tx\| = \|x\|.$$

Methods

The step of the research work is to determine the value of duality product of ℓ^p and determine symmetry of the duality product. Then using Theorem 3, it will be shown that ℓ^p with $p = 2$ isomorphic with Hilbert space.

Results and Discussion

The normed space ℓ^p with $p \geq 2$ is a complete normed space or called Banach space and has been written on (Kreyszig, 1978). Furthermore, this writing will focus on ℓ^p with $p = 2$, because that for $p > 2$, ℓ^p is not an inner product space.

Define ℓ^{p*} is dual of ℓ^p, such that there is T which is a linear operator $T: \ell^p \rightarrow \ell^{p*}$. Duality product of the space is $\langle Tx, x\rangle$ for $\forall x \in \ell^p$ (Molnár & Šemrl, 2012). Because T is a linear operator, the following applies

$$T(x + y) = T(x) + T(y)$$

$$T(\alpha x) = \alpha T(x)$$

for $\forall x \in X$ and α elements of field. In (Casazza, Fickus, Mixon, & Peterson, 2013; Drivaliaris & Yannakakis, 2005), it is said that consequence of Theorem 3 is as follows

A real Banach space X is isomorphic to a Hilbert space if and only if there is an isomorphism $T: X \rightarrow X'$ that is positive and symmetric.

In order to prove that ℓ^pisomorphic to Hilbert space, it will be proven that the values of $\langle Tx, x\rangle \geq 0$ and $\langle Tx, y\rangle = \langle Ty, x\rangle$ for $\forall x, y \in \ell^p$.

Proof for T is a positive operator.

Using the definition of dual space, that ℓ^{p*} is dual of ℓ^p, then $\forall x \in \ell^{p*}$ applies the norm property, i.e.

$$(N1)\ \|x\| \geq 0$$

$$(N2)\ \|x\| = 0 \leftrightarrow x = 0$$

$$(N3)\ \|\alpha x\| = |\alpha|\|x\|$$

$$(N4)\ \|x + y\| \leq \|x\| + \|y\|.$$

Based on these properties, it is obtained that $\forall Tx \in \ell^{p*}, x \in \ell^p$, Tx satisfies $(N1) - (N4)$.

It will be proven that $\langle Tx, x\rangle \geq 0$. Because $\langle Tx, x\rangle$is an inner product, so $\langle Tx, x\rangle$ satisfies

$$(DP1)\ \langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

$$(DP2)\ \langle \alpha x, y\rangle = \alpha\langle x, y\rangle$$

$$(DP3)\ \langle x, y\rangle = \langle y, x\rangle$$

$$(DP4)\quad \begin{array}{c} \langle x, x\rangle \geq 0 \\ \langle x, x\rangle = 0 \leftrightarrow x = 0 \end{array}$$

Since $\langle\cdot,\cdot\rangle: \ell^{p*} \times \ell^{p} \to F$ and Tx are operator on ℓ^p, then the value of $\langle Tx, x\rangle$ will satisfy (DP4). As a result value of $\langle Tx, x\rangle \geq 0$.

Proof for T is a symmetric operator.

It will be proven that $\langle Tx, y\rangle = \langle Ty, x\rangle$. Because T is an isomorphism of ℓ^p onto ℓ^{p*}, consequently $\|Tx\| = \|x\|$, so

$$\langle Tx, y\rangle = \langle \|Tx\|, \|y\|\rangle$$

$$= \langle \|y\|, \|Tx\|\rangle = \langle y, Tx\rangle.$$

The conclusion of the explanation is that T is an isomorphism of ℓ^p onto ℓ^{p*}, so that ℓ^p is isomorphic to the Hilbert space.

Conclusion

The conclusion of this paper is determination of a ℓ^p space as the space Hilbert can use the dual of ℓ^p space isomorphic with it, so that using the dual, there will be a lot of other ℓ^p space that will be proved easily as Hilbert space. Based on the proof, the data that is vector space can be isomorphic with Hilbert space using dual of vector space.

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